# Hypersonic Laminar Boundary-Layer Flow over an Expansion Corner

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## Nomenclature

$f$ = velocity profile defined by Eq. (1)
$g$ = total enthalpy profile defined by Eq. (2)
$H$ = total enthalpy
$L$ = characteristic length, defined as the length from the leading edge to the corner
$M$ = Mach number
$p$ = pressure
$Pr$ = Prandtl number
$R$ = $(p_e/p_\infty)(\rho_\infty/\rho_e)^2(u_\infty/u_e)$
$T$ = temperature
$u$ = velocity of the flow along the surface
$x$ = distance along the surface
$\gamma$ = ratio of specific heats
$\delta$ = boundary-layer thickness in $(x,y)$ plane
$\bar{\delta}$ = boundary-layer thickness defined by $\int_0^\delta \rho/\rho_e \, dy$
$\delta^*$ = boundary-layer displacement thickness in $(x,y)$ plane
$Re_\infty$ = Reynolds number at the corner
$\xi$ = coordinate defined by $x/L$
$\eta$ = coordinate defined by $1/\bar{\delta} \int_0^y \rho/\rho_e \, dy$
$\phi_w$ = corner turning angle of the surface

*Subscripts*

$c$ = corner
$e$ = edge of boundary layer
$0$ = stagnation point
$w$ = surface
$\infty$ = freestream conditions

## Theme

IN the analysis of the interaction of a hypersonic laminar boundary layer with a corner-expansion wave, Sullivan[1] has presented a simple method of solution based on the cold-wall similarity model. This method is self-consistent, since the calculations confirm the basic assumptions of cold-wall similarity in hypersonic flow, and it can be regarded as a first-order approximation to a local-similarity solution. The method is based on the fact that the boundary-layer displacement effects become very significant, and consequently the pressure-gradient effects generally dominate the boundary-layer-growth behavior in a hypersonic flow. However, using the cold-wall similarity assumption suggested by Lees,[2] the formulation of the interaction problem can be simplified greatly.

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Under the cold-wall similarity and $u_e = u_\infty$ assumptions, the four partial differential equations of the problem were reduced to two ordinary differential equations by Sullivan. In this work, the assumption that $u_e = u_\infty$ is removed, and the local similarity result for the total enthalpy profile is used in the energy equation. This improvement will be significant for the case of supersonic flow, where the validity of Sullivan's model is questionable.

The major features of the flow at the edge of the boundary layer are predicted successfully, and approximate flow profiles downstream of the corner are obtained easily. The initial conditions at the corner involving the pressure ratio and the boundary-layer-displacement thickness are discussed. The results are compared with other methods[1,3] and existing experimental data.[4]

The analysis gives some physical insight into the problem. The present method is applied to a hypersonic flow over a cold surface. Any upstream influence is neglected, since the boundary layer is supercritical. For small turning angles, the centrifugal effects are not important and are omitted.

## Contents

The difference between cold-wall similarity and local similarity methods was explained clearly in Refs. 1 and 2. For local similarity, the velocity and total enthalpy profiles are assumed to be functions of the local pressure gradient and the ratio of the density at the edge of the boundary layer to that at the surface. However, if the surface temperature is much lower than that at the edge of the boundary layer, the influence of the pressure gradient on the velocity profile will be small according to qualitative physical arguments in the momentum equation.

In the integral method, the profiles of velocity and total enthalpy in laminar compressible boundary-layer flow can be

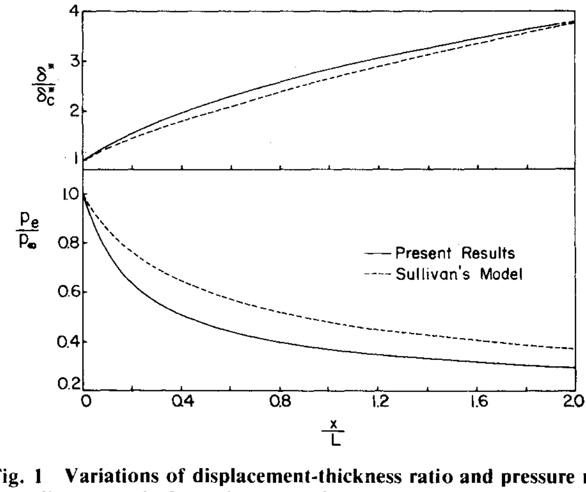


**Fig. 1 Variations of displacement-thickness ratio and pressure ratio with distance ratio from the corner for the case of $g_w = 0.2$, $\phi_w = 5°$, $Re_\infty = 1.644 \times 10^5$, $M_\infty = 10$, $Pr = 1$.**

expressed by

$$f = \sum_{n=0}^{6} a_n(\xi)\eta^n \quad (1)$$

$$g = \sum_{n=0}^{7} b_n(\xi)\eta^n \quad (2)$$

We note that, for the local similarity approximation, $a_n$ and $b_n$ are dependent on the local-pressure gradient and, therefore, dependent on $\xi$. For the cold-wall similarity assumption, $a_n$ are independent of $\xi$ and $b_n$ still are dependent on $\xi$. Sullivan has shown that, even if the cold-wall assumption does not apply and the pressure-gradient term in the momentum equation is retained, its contribution is small for favorable pressure gradients. Hence the cold-wall similarity concept should be applicable over a wider range of surface temperatures.

Under the cold-wall similarity assumption, the approximate velocity profile $f$ is determined first from the boundary conditions. The integral form of the momentum equation yields the following relation between the pressure gradient and the growth of the boundary-layer thickness

$$\frac{d}{d\xi}\left(\frac{p_e}{p_\infty}\right) = \frac{p_e/p_\infty \gamma M_e^2}{\lambda[(1-M_e^2)F_1 - E_1 + E_2]} \times \left[(\sqrt{\lambda Re_\infty}\, F_1 \frac{d}{d\xi}\left(\frac{\delta}{L}\right) - 2R\right] \quad (3)$$

where $F_1 = 985/9009$, $E_1 = 5450/9009$, $E_2 = (G-E_1)T_0/T_e + E_1$, $G = (1+g_w)/2 + 3b_1/28 + b_2/42$, and $\lambda = Re_\infty(\delta/L)^2$.

The total-enthalpy profile parameter $b_1$ can be calculated directly from the integral form of the energy equation. However, the effect of the pressure gradient on the total-enthalpy profile is much smaller than that due to the growth of the boundary-layer thickness. The local value of $b_1$ can be obtained approximately from the energy equation

$$b_1 = \frac{\dfrac{31}{242}(1-g_w) - \dfrac{151}{9009}\dfrac{u_e^2}{H_e}(1-Pr)}{\dfrac{1}{Pr\,\lambda_0} + \dfrac{821}{24{,}024}} \quad (4)$$

where $\lambda_0 = 36{,}036/985$.

In obtaining the solution for the total-enthalpy profile, Sullivan has assumed $u_e = u_\infty$ in the energy equation in order to simplify the calculation. However, Eq. (4) corresponds to the energy equation obtained by Sullivan without using the hypersonic assumption $u_e = u_\infty$.

The relation between the growth of the boundary-layer thickness and the growth of the displacement thickness is obtained from the definition

$$\frac{d}{d\xi}\left(\frac{\delta}{L}\right) = \frac{1}{J}\frac{d}{d\xi}\left(\frac{\delta^*}{L}\right) \quad (5)$$

where $J = (G-E_1)T_0/T_e - F_1$, and $\delta^*$ is the displacement thickness.

The solution of the physical problem then is reduced to integrating Eqs. (3) and (5) simultaneously by using Eq. (4) and the relation between the local slope of the effective body surface and the Prandtl-Meyer deflection angle. The differences between the present analysis and Sullivan's method are 1) here $u_e \neq u_\infty$; and 2) Sullivan assumed that $J \approx T_0/T_\infty (G-E_1)_c$ because of the hypersonic assumption.

By assuming that the upstream-influence effects for locally hypersonic flow are small, the initial condition for $p_e/p_\infty$ is taken as unity. Two kinds of initial conditions for $\delta/L$ are considered: 1) the initial value is obtained[1] from the solution just upstream of the corner, $\delta/L = (\lambda_0/Re_\infty)^{1/2}$; and 2) for a larger turning angle, $\delta/L$ is obtained approximately from the flow geometry as $\delta/L \approx (\lambda_0/Re_\infty)^{1/2}(1+\tan\phi_w \tan\Delta\theta)$, where $\phi_w$ is the corner turning angle and $\Delta\theta = \phi_w/2 + \tan^{-1}(d\delta^*/dx)_c$.

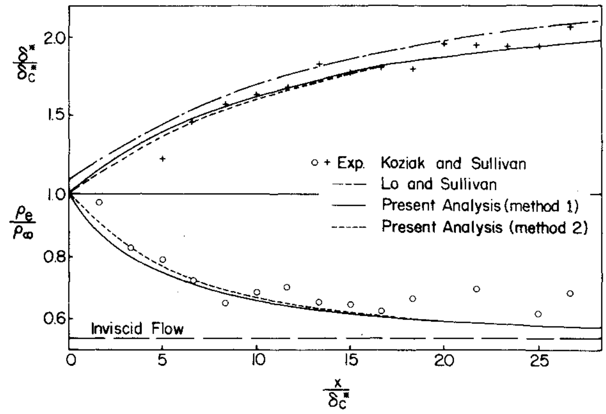


**Fig. 2 Variations of displacement-thickness ratio and density ratio with distance ratio from the corner for the case of $g_w = 0.22$, $\phi_w = 5°$, $Re_\infty = 8\times10^5$, $M_\infty = 6.5$, and $Pr = 0.7$.**

The approximate results for the velocity and total enthalpy profiles used in the local-similarity model can be obtained from the solutions for $p_e/p_\infty$ and $\delta/L$ from Eqs. (3) and (5), along with the following relations

$$a_1 = 2 - (12 + b_1/g_w)a_2/30 \quad (6a)$$

$$a_2 = -\frac{1}{2R}\frac{\rho_e}{\rho_w}\frac{\lambda}{u_e}\frac{du_e}{d\xi} \quad (6b)$$

$$b_2 = (u_e^2/2H_e)(1-Pr)a_1^2 \quad (6c)$$

$$b_3(u_e^2/2H_e)(1-Pr)a_1 a_2 \quad (6d)$$

Figure 1 shows the difference between the present analysis and Sullivan's method for the case $g_w = 0.2$, $\phi_w = 5°$, $Re_\infty = 1.644\times10^5$, $M_\infty = 10$, and $Pr = 1$. The initial condition for $\delta/L$ is calculated by method 1. It is seen that a small change in the displacement-thickness ratio results in a significant change in the pressure ratio. Figure 2 shows the results compared with those obtained from the integral method[3] of the simplified Navier-Stokes equation and experimental data.[4] The experiments show no evidence that the upstream effect is significant for the present conditions, whereas Lo and Sullivan have predicted a significant upstream effect on the pressure and displacement thickness ratios at the corner in their calculations. In summary, it can be stated that the present method provides a more accurate treatment of the boundary-layer equations than that suggested by Sullivan.

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